SPECIAL FUNCTIONS IN RADIATIVE TRANSFER AND THEIR PROPERTIES: $Cis_n(x,\theta)$ AND $Dis_n(x)$

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1. $Cis_n(x,\theta)$ and $Dis_n(x)$ Functions

The $Cis_n(x,\theta)$ functions are defined as [1]

$$Cis_n(x,\theta) = \int_0^\theta Ki_n(x\sec\alpha)\ \sin\alpha\ (\cos\alpha)^{n-3}\ \mathrm{d}\alpha \tag{1.1}$$

where Ki_n is the nth order Bickley-Naylor functions and they appear in radiative integral transfer equations of rectangular geometry and they satisfy following differentiation relation

$$\frac{d}{dx}Cis_{n+1}(x,\theta) = -Cis_n(x,\theta) \tag{1.2}$$

and the integration relations

$$Cis_{n+1}(x,\theta) = \int_x^\infty Cis_n(t,\theta)dt \tag{1.3}$$

Cis_n functions are found rather simpler in terms of functions Dis_n's. The Dis_n functions have the following general form:

$$\frac{Dis_n(x)}{x^{n-1}} = \frac{1}{(n-1)!}\int_1^\infty (t-1)^{n-1}\frac{Ki_2(xt)}{t}dt,\ n\geq 1 \tag{1.4}$$

By definition, Dis_0 is

$$Dis_0(x) = \frac{Ki_2(x)}{x} \tag{1.5}$$

The series expansions for Dis_1 and Dis_2 are found as

$$\begin{aligned} Dis_1(x) &= \int_x^\infty \frac{Ki_2(t)}{t}dt = x\int_x^\infty \frac{K_0(t)}{t^2}dt = \frac{\pi}{2}x - \left[1+\gamma+\ln\left(\frac{x}{2}\right)\right] \\ &+ \left[\gamma+\ln\left(\frac{x}{2}\right)\right]\sum_{k=1}^\infty \frac{(x^2/4)^k}{(k!)^2(2k-1)} - \sum_{k=1}^\infty \left[\frac{1+(2k-1)\Phi(k+1)}{(k!)^2(2k-1)^2}\right]\left(\frac{x^2}{4}\right)^k \end{aligned} \tag{1.6}$$

and

$$Dis_2(x) = \frac{\pi}{4}(1-x^2) + x\left[\gamma + \ln\left(\frac{x}{2}\right)\right] - x\left[\gamma + \ln\left(\frac{x}{2}\right)\right]\sum_{k=1}^{\infty}\frac{(x^2/4)^k}{(k!)^2(4k^2-1)}$$

$$+x\sum_{k=1}^{\infty}\frac{(x^2/4)^k}{(k!)^2(4k^2-1)(2k+1)} + x\sum_{k=1}^{\infty}\left[\frac{1+(2k-1)\Phi(k+1)}{(k!)^2(4k^2-1)(2k-1)}\right]\left(\frac{x^2}{4}\right)^k \tag{1.7}$$

where γ is the Euler constant and

$$\Phi(k+1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \tag{1.8}$$

This new function, also, satisfies the relations

$$\frac{d}{dx}Dis_{n+1}(x) = -Dis_n(x) \tag{1.9}$$

and

$$Dis_{n+1}(x) = \int_x^{\infty} Dis_n(t)dt \tag{1.10}$$

A recurrence relation which involve Bickley functions exists to compute high order functions.

$$nDis_{n+1}(x) = Ki_{n+2}(x) - xDis_n(x),\ n \geq 1 \tag{1.11}$$

The asymptotic expansions for Dis_n's are found to be

$$Dis_n(x) \approx \sqrt{\frac{\pi}{2x}}\frac{e^{-x}}{(x+n)}\left\{1 - \frac{(9+4n^2)}{8x} + \frac{4(111n^2-65n+86)}{2!(8x)^2}\right\} \tag{1.12}$$

In the light of these newly defined functions, Cis_n lead to

$$Cis_{n+1}(x,\theta) = Dis_n(x) - (\cos\theta)^{n-1}Dis_n(x\sec\theta) \tag{1.13}$$

References

[1] Z. Altaç, Integrals involving Bickley and Bessel functions in radiative transfer, and generalized Exponential Integral functions. *ASME J. Heat Transfer*. **118** 789-792 (1996).

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