SPECIAL FUNCTIONS IN RADIATIVE TRANSFER AND THEIR PROPERTIES : $Bis_n(x,\theta)$

By Prof. Dr. Zekeriya ALTAÇ*

$Bis_n(x,\theta)$ FUNCTIONS

Integral form of $Bis_n(x,\theta)$ function is [1]

$$Bis_n(x,\theta) = \int_{\alpha=0}^{\theta} Ki_n(x\sec\alpha)\cos^{n-2}\alpha d\alpha \tag{1}$$

where Ki_n is the Bickley-Naylor function of the nth order. This function appears in radiative integral transfer equations of rectangular geometry and they satisfy following differentiation and integration rules

$$\frac{dBis_{n+1}}{dx} = -Bis_n(x) \qquad \text{and} \qquad Bis_{n+1}(x) = \int_x^{\infty} Bis_n(x')\mathrm{d}x' \tag{2}$$

For $-\pi/2 < \theta < \pi/2$, the series expansions for $Bis_1(x,\theta)$, $Bis_2(x,\theta)$ and $Bis_3(x,\theta)$ are obtained as

$$\begin{aligned} Bis_1(x,\theta) &= x\theta + \frac{\pi}{2}\ln(\sec\theta+\tan\theta) + x\tan\theta\left[\gamma - 2 + \ln(\frac{x}{2}\sec\theta)\right] \\ &+ x\tan\theta\left[\gamma + \ln(\frac{x}{2}\sec\theta)\right]\sum_{k=1}^{\infty}\frac{(x/2)^{2k}A_k(\tan\theta)}{(k!)^2(2k+1)^2} \\ &+ \left(\frac{x\tan\theta}{2}\right)\left\{\sum_{k=1}^{\infty}\frac{(x/2)^{2k}A_k(\tan\theta)}{k!(k+1)!(2k+1)} - \sum_{k=1}^{\infty}\frac{(\sec\theta)^{2k}+4k^2C_{k-1}(\tan\theta)}{k(k!)^2(2k+1)^2}\left(\frac{x}{2}\right)^{2k}\right. \\ &\left. - \sum_{k=1}^{\infty}\frac{\left[4k^2+2k-1+2k(2k+1)(k+1)\Phi(k+1)\right]}{kk!(k+1)!(2k+1)^2}\left(\frac{x}{2}\right)^{2k}A_k(\tan\theta)\right\} \end{aligned} \tag{3}$$

* Osmangazi University, School of Engineering and Architecture, Mechanical Engineering Department, 26480 Bati Meselik, Eskisehir, TURKEY
E-mail address: **zaltac@ogu.edu.tr**

$$Bis_2(x,\theta)=\theta\left(1-\frac{x^2}{2}\right)-x\frac{\pi}{2}\ln(\sec\theta+\tan\theta)-\frac{x^2\tan\theta}{2}\left\{\left[\gamma-\frac{5}{2}+\ln(\frac{x}{2}\sec\theta)\right]+\right.$$
$$+\left[\gamma+\ln(\frac{x}{2}\sec\theta)\right]\sum_{k=1}^{\infty}\frac{(x/2)^{2k}A_k(\tan\theta)}{k!(k+1)!(2k+1)}$$
$$-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\left[4k^2+2k-1+2k(2k+1)(k+1)\Phi(k+1)\right]}{k(k+1)!(2k+1)^2}\left(\frac{x}{2}\right)^{2k}A_k(\tan\theta) \tag{4}$$
$$\left.-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(\sec\theta)^{2k}+4k^2C_{k-1}(\tan\theta)}{kk!(k+1)!(2k+1)^2}\left(\frac{x}{2}\right)^{2k}\right\}$$

and

$$Bis_3(x,\theta)=\frac{\pi}{4}\sin\theta-\left(x-\frac{x^3}{6}\right)\theta+\frac{\pi x^2}{4}\ln(\sec\theta+\tan\theta)+\frac{x^3\tan\theta}{6}\left[\gamma-\frac{17}{6}+\ln(\frac{x}{2}\sec\theta)\right]$$
$$+\frac{x^3\tan\theta}{2}\left\{\left[\gamma+\ln(\frac{x}{2}\sec\theta)\right]\sum_{k=1}^{\infty}\frac{(x/2)^{2k}A_k(\tan\theta)}{k!(k+1)!(2k+1)(2k+3)}\right.$$
$$-\sum_{k=1}^{\infty}\frac{\left[4k^2+2k-1+2k(2k+1)(k+1)\Phi(k+1)\right]}{k(k+1)!^2(2k+1)^2(2k+3)}\left(\frac{x}{2}\right)^{2k}A_k(\tan\theta) \tag{5}$$
$$-\sum_{k=1}^{\infty}\frac{(x/2)^{2k}A_k(\tan\theta)}{k!(k+1)!(2k+1)(2k+3)^2}$$
$$\left.-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(\sec\theta)^{2k}+4k^2C_{k-1}(\tan\theta)}{kk!(k+1)!(2k+1)^2(2k+3)}\left(\frac{x}{2}\right)^{2k}\right\}$$

where

$$\Phi(k+1)=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k} \tag{6}$$

and γ is the Euler constant. A_n and C_n's in satisfy the following recurrence relations:

$$A_n(x)=\frac{1}{(2n+1)}\left[(1+x^2)^n+2nA_{n-1}(x)\right],\qquad n\geq 1 \tag{7}$$
$$C_n(x)=\frac{1}{2n(2n+1)}\left[(1+x^2)^n-A_n(x)+4n^2C_{n-1}(x)\right],\qquad n\geq 1 \tag{8}$$

The first few orders of the polynomial A_n are

$$A_0(x)=1,\qquad A_1(x)=\frac{1}{3}(x^2+3),\qquad A_2(x)=\frac{1}{15}(3x^4+10x^2+15),\ldots \tag{9}$$

and those of C_n are

$$C_0(x) = 1 - \frac{\tan^{-1} x}{x}, \qquad C_1(x) = \frac{1}{9}(x^2 + 6) - \frac{2}{3}\frac{\tan^{-1} x}{x},$$
$$C_2(x) = \frac{1}{225}(9x^4 + 35x^2 + 120) - \frac{8}{15}\frac{\tan^{-1} x}{x}, \cdots \tag{10}$$

As a special case, for $\theta=\pi/2$, Bis_n functions yield in terms of exponential integral functions as

$$Bis_n(x, \frac{\pi}{2}) = \frac{\pi}{2} E_n(x) \tag{11}$$

REFERENCES

[1] Z. Altaç, Integrals involving Bickley and Bessel functions in radiative transfer, and generalized Exponential Integral functions. *ASME J. Heat Transfer*. 118 789-792 (1996).
[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, pp. 483, Dover Publications Inc., (1972).
[3] E. E. Lewis and W. F. Miller, *Computational Methods of Neutron Transport*, John Wiley Sons, 1984.