THE BICKLEY-NAYLOR FUNCTIONS Ki$_n$(x) AND THEIR PROPERTIES:

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Various integral forms of the Bickley-Naylor functions $Ki_n(x)$ are listed as [1-3]

$$Ki_n(x) = \int_0^\infty \frac{\exp(-x\cosh t)}{\cosh^n t} dt \tag{1}$$

$$Ki_n(x) = \frac{1}{(n-1)!}\int_x^\infty (t-x)^{n-1} K_0(t) dt \tag{2}$$

$$\frac{Ki_n(x)}{x^n} = \frac{1}{(n-1)!}\int_1^\infty (t-1)^{n-1} K_0(xt) dt \tag{3}$$

$$Ki_n(x) = \int_1^\infty \frac{\exp(-xt)dt}{t^n\sqrt{t^2-1}} \tag{4}$$

$$Ki_n(x) = \int_0^{\pi/2} \exp(-x/\cos\alpha)\cos^{n-1}\alpha d\alpha \tag{5}$$

$$Ki_n(x) = \int_0^{\pi/2} \exp(-x/\sin\alpha)\sin^{n-1}\alpha d\alpha \tag{6}$$

and $Ki_0(x) = K_0(x)$ where $K_0(x)$ is the modified Bessel function of the zeroth order.

The series expansion forms of the first and the second order Bickley functions are given as

$$Ki_1(x) = \frac{\pi}{2} + x\left[\gamma + \ln(x/2)\right]\sum_{k=0}^{\infty}\frac{(x/2)^{2k}}{(k!)^2(2k+1)} - x\sum_{k=0}^{\infty}\frac{(x/2)^{2k}}{(k!)^2(2k+1)^2} - x\sum_{k=0}^{\infty}\frac{(x/2)^{2k}\Phi(k+1)}{(k!)^2(2k+1)} \tag{7}$$

and

$$Ki_2(x) = 1 - \frac{\pi}{2}x - \frac{x^2}{2}\left[\gamma + \ln(x/2)\right]\sum_{k=0}^{\infty}\frac{(x/2)^{2k}}{k!(k+1)!(2k+1)} + \frac{x^2}{4}\sum_{k=0}^{\infty}\frac{(4k+3)(x/2)^{2k}}{[(k+1)!(2k+1)]^2} + \frac{x^2}{2}\sum_{k=0}^{\infty}\frac{(x/2)^{2k}\Phi(k+1)}{k!(k+1)!(2k+1)} \tag{8}$$

where

$$\Phi(k+1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \tag{9}$$

and γ is the Euler's constant ($\gamma = 0.57721566490153286$).

The Bickley functions also satisfy the following recurrence relation [1].

$$nKi_{n+1}(x) = (n-1)Ki_{n-1}(x) - xKi_n(x) + xKi_{n-2}(x), \quad n \geq 2 \tag{10}$$

The asymptotic expansions of Bickley functions are given as [2],

$$Ki_n(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}\left\{1 - \frac{(4n+1)}{8x} + \frac{3(6n^2+24n+3)}{2!(8x)^2} - \cdots\right\} \tag{11}$$

I have obtained the following improved formulae for the asymptotic expansion:

$$Ki_n(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}\begin{cases} 1 - \frac{(m+1)}{8x} + \frac{3(m^2+6m+3)}{2!(8x)^2} - \frac{15(m+5)(m^2+10m+3)}{3!(8x)^3} \\ + \frac{105(m^4+28m^3+242m^2+644m+105)}{4!(8x)^4} \\ - \frac{945(m+9)(m^4+36m^3+386m^2+1116m+105)}{5!(8x)^5} + \ldots \end{cases} \tag{12}$$

where m=4n.

I have also developed the general series expansion formulae for the n'th order Bickley-Naylor functions as follows:

$$Ki_n(x) = 2^{n-2}\sum_{k=0}^{n-1}\frac{(-x/2)^k}{k!(n-k-1)!}\Gamma^2\left(\tfrac{n-k}{2}\right) + (-x)^n\sum_{k=0}^{\infty}\frac{(x/2)^{2k}(2k)!}{(k!)^2(n+2k)!}\left(\Phi(k+1) - \Phi(2k+1) + \Phi(2k+n+1) - \gamma - \ln(\tfrac{x}{2})\right) \tag{13}$$

where $\Gamma(k)$ is the Gamma function.

The Bickley functions satisfy following differentiation and integration rules

$$\frac{d}{dx}Ki_{n+1}(x) = -Ki_n(x) \quad \text{and} \quad Ki_{n+1}(x) = \int_x^{\infty} Ki_n(x')\,dx' \tag{14}$$

Repeated differentiation yields

$$\frac{d^n}{dx^n}\left(Ki_n(x)\right) = (-1)^n K_0(x) \tag{15}$$

An integral relation between the exponential integral functions and Bickley-Naylor functions of n'th order is as follows:

$$E_n(x) = \frac{2}{\pi}\int_0^{\pi/2} Ki_n(x\sec\theta)\cos^{n-2}\theta\,d\theta \tag{16}$$

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, pp. 483, Dover Publications Inc., (1972).

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[3] E. E. Lewis and W. F. Miller, *Computational Methods of Neutron Transport*, John Wiley Sons, 1984.