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# Exact series expansions, recurrence relations, properties and integrals of the generalized exponential integral functions

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**Abstract**

Generalized exponential integral functions (GEIF) are encountered in multi-dimensional thermal radiative transfer problems in the integral equation kernels. Several series expansions for the first-order generalized exponential integral function, along with a series expansion for the general $n$th order GEIF, are derived. The convergence issues of these series expansions are investigated numerically as well as theoretically, and a recurrence relation which does not require derivatives of the GEIF is developed. The exact series expansions of the two dimensional cylindrical and/or two-dimensional planar integral kernels as well as their spatial moments have been explicitly derived and compared with numerical values.


*Keywords:* Generalized exponential integral functions; Multidimensional radiative transfer; Anisotropic scattering; Cylindrical medium

## 1. Introduction

Generalized exponential integral functions (GEIF) arise in the study of thermal radiative transfer of two-dimensional planar and two-dimensional cylindrical ($r$–$z$) media. The scope of these functions extends from nuclear reactor physics to astrophysics in a wide range of theoretical and applied sciences [1–19]. The kernels and the surface irradiation terms of the radiative integral equations, in two-dimensional cylindrical media, involve integrals of the generalized exponential integral functions [13,16,17]. Several series expansions have been proposed for the evaluation of the first-order GEIF [11,19–21]. An asymptotic expansion for the first-order generalized exponential integral function (for $x \gg 1$) was derived by Breig and Crosbie [11], and a recurrence relation was given by Yuen and Yong [18]. A series expansion for the $n$th-order GEIF, in terms of the exponential integral functions, was recently developed [21].

The purpose of this paper is to obtain exact series expansions of the GEIFs and their integrals of interest in thermal radiative transfer in a fashion which is simple, accurate and adaptable to fast computation.

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**Nomenclature**

$\mathbb{E}_{n,v}(a,b)$, function defined by Eq. (35)
$E_n(x)$, $n$th-order exponential integral function
$F(\alpha,\beta;\gamma;x)$, hypergeometric function defined by Eq. (6)
${}_1F_2(\alpha;\beta,\gamma;x)$, generalized hypergeometric function defined by Eq. (37)
$I_n(x)$, $K_n(x)$, $n$th-order modified Bessel functions
$I_{n/2}(x)$, $K_{n/2}(x)$, modified Bessel functions fractional orders
$J_n(x)$, $n$th-order Bessel function
$Ki_n(x)$, $n$th-order Bickley–Naylor function
$N$, the number of terms used in a series
$a_k$, general term of an infinite series

*Greek symbols*

$\mathscr{E}_n(x,y)$ $n$th order general exponential integral function defined by Eqs. (1) and (2)
$\Gamma(x)$ Gamma function
$\mathscr{K}^{v,\mu}_{\alpha,n}(a,b,c)$ function defined by Eq. (40)
$\psi(j+1)$ function defined as $\psi(j+1)=-\gamma+\sum_{n=1}^{j}(1/n)$
$\gamma$ Euler constant
$\eta$ defined as $x\sqrt{1+y^2}$
$\lambda$ defined as $\sqrt{a^2+b^2}$
$\rho$ defined as $\rho=\alpha+\mu+v$ for Eqs. (47a) and (47b)
$\zeta$ convergence criterion

## 2. Derivation

The $n$th-order generalized exponential integral functions, $\mathscr{E}_n(x,y)$, have the following general form [19]:

$$\mathscr{E}_n(x,y)=\frac{1}{(n-1)!}\int_{t=x}^{\infty}(t-x)^{n-1}\frac{\exp\left[-\sqrt{t^2+(xy)^2}\right]}{\sqrt{t^2+(xy)^2}}\,\mathrm{d}t, \tag{1}$$

Eq. (1), upon a change of variables, could also be rewritten as:

$$\frac{\mathscr{E}_n(x,y)}{x^{n-1}}=\frac{1}{(n-1)!}\int_{t=1}^{\infty}(t-1)^{n-1}\frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}\,\mathrm{d}t, \tag{2}$$

The zeroth-order exponential integral function is defined as $\mathscr{E}_0(x,y)=\mathrm{e}^{-\eta}/\eta$ [19] where $\eta=x\sqrt{1+y^2}$ definition has been used throughout this article to save space.

It is possible to obtain several forms of series expansions for the GEIFs. First, the series expansions for $\mathscr{E}_1(x,y)$ will be derived and discussed, then the generalization to the $n$th-order GEIFs, $\mathscr{E}_n(x,y)$, will be given. To begin the derivation, let us recall the following integral form of the zeroth-order modified Bessel function [22]:

$$K_0(xy)=\int_{t=0}^{\infty}\frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}\,\mathrm{d}t, \tag{3}$$

Rearranging the integral intervals of Eq. (2) for $n=1$ and using Eq. (3), we obtain

$$\mathscr{E}_1(x,y)=\int_{t=1}^{\infty}\frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}\,\mathrm{d}t=K_0(xy)-\int_{t=0}^{1}\frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}\,\mathrm{d}t, \tag{4}$$

To evaluate the integral on the right-hand side (RHS) of Eq. (4), the exponential term is expanded to a Taylor series. Upon employing a change of variable $t=\sqrt{u}$ and reorganizing the resulting terms, the integral yields:

$$\begin{aligned}\int_{t=0}^{1}\frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}\,\mathrm{d}t &= \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\int_{t=0}^{1}\left(\sqrt{t^2+y^2}\right)^{k-1}\mathrm{d}t\\ &=\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-x)^k y^{k-1}}{k!}\int_{u=0}^{1}u^{-1/2}\left(1+\frac{u}{y^2}\right)^{(k-1)/2}\mathrm{d}u\\ &=\frac{1}{y}\sum_{k=0}^{\infty}\frac{(-xy)^k}{k!}F\left(\frac{1}{2},\frac{1-k}{2};\frac{3}{2};\frac{-1}{y^2}\right)\end{aligned} \tag{5}$$

where $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function and is defined as [22]

$$\begin{aligned}F(\alpha,\beta;\gamma;z)&=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_{t=0}^{1}t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,\mathrm{d}t\\ &=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)\Gamma(\beta+k)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+k)}\frac{x^k}{k!},\end{aligned} \tag{6}$$

where $\Gamma(x)$ is the Gamma function. Thus, substituting the outcome of Eq. (5) into Eq. (4), we arrive at the first series expansion for $\mathscr{E}_1(x,y)$:

$$\mathscr{E}_1(x,y)=K_0(xy)-\frac{1}{y}\sum_{k=0}^{\infty}\frac{(-xy)^k}{k!}F\left(\frac{1}{2},\frac{1-k}{2};\frac{3}{2};-\frac{1}{y^2}\right), \tag{7}$$

To obtain a second series expansion for $\mathscr{E}_1(x,y)$, we will make use of the following integral representation for the exponential term of Eq. (2) [23]:

$$\frac{\exp[-x\sqrt{t^2+y^2}]}{\sqrt{t^2+y^2}}=\int_{u=0}^{\infty}uJ_0(ut)\frac{\exp[-y\sqrt{x^2+u^2}]}{\sqrt{x^2+u^2}}\,\mathrm{d}u, \tag{8}$$

where $J_0(x)$ is the zeroth-order Bessel function.

Having set $n=1$ in Eq. (2), we substitute Eq. (8) into Eq. (4). Changing the order of integration variables, first we perform integration over $t$-variable:

$$\begin{aligned}\mathscr{E}_1(x,y)&=\int_{t=1}^{\infty}\int_{u=0}^{\infty}uJ_0(ut)\frac{\exp[-y\sqrt{x^2+u^2}]}{\sqrt{x^2+u^2}}\,\mathrm{d}u\,\mathrm{d}t\\ &=\int_{u=0}^{\infty}\left\{\int_{t=1}^{\infty}uJ_0(ut)\,\mathrm{d}t\right\}\frac{\exp[-y\sqrt{x^2+u^2}]}{\sqrt{x^2+u^2}}\,\mathrm{d}u,\end{aligned} \tag{9}$$

To complete the integration of Eq. (9), we recall the following identities [23]:

$$\int_{t=0}^{a}J_\nu(ut)\,\mathrm{d}t=\frac{2}{u}\sum_{k=0}^{\infty}J_{\nu+2k+1}(au),\quad (\nu>-1) \tag{10}$$

and

$$\int_{t=0}^{\infty}J_0(ut)\,\mathrm{d}t=\frac{1}{u} \tag{11}$$

Recasting the $t$-integration of Eq. (9) as two integrals, similar to Eq. (4), over $(0, \infty)$ and $(0, 1)$ intervals to use Eqs. (10) and (11), Eq. (9) yields the following form:

$$\begin{aligned}\mathscr{E}_1(x,y) &= \int_{u=0}^{\infty}\left\{1-2\sum_{n=0}^{\infty}J_{2n+1}(u)\right\}\frac{\exp\left[-y\sqrt{x^2+u^2}\right]}{\sqrt{x^2+u^2}}\,\mathrm{d}u \\ &= \int_{u=0}^{\infty}\frac{\exp\left[-y\sqrt{x^2+u^2}\right]}{\sqrt{x^2+u^2}}\,\mathrm{d}u-2\sum_{n=0}^{\infty}\int_{u=0}^{\infty}J_{2n+1}(u)\frac{\exp\left[-y\sqrt{x^2+u^2}\right]}{\sqrt{x^2+u^2}}\,\mathrm{d}u,\end{aligned} \tag{12}$$

The first integral of Eq. (12), via Eq. (3), yields $K_0(xy)$. At this stage, for the second integral on the RHS, we have two alternative paths to follow which will yield two different series expansions. One of the alternatives is to use the following identity (for $\alpha>0$, $\beta>0$ and $\gamma>0$) [23]:

$$\int_0^{\infty}\frac{J_v(\gamma x)\exp\left[-\alpha\sqrt{x^2+\beta^2}\right]}{\sqrt{x^2+\beta^2}}\,\mathrm{d}x = I_{v/2}\left(\frac{\beta}{2}\left[\sqrt{\alpha^2+\gamma^2}-\alpha\right]\right)K_{v/2}\left(\frac{\beta}{2}\left[\sqrt{\alpha^2+\gamma^2}+\alpha\right]\right), \tag{13}$$

where $I_{v/2}(x)$ and $K_{v/2}(x)$ are the modified Bessel functions of fractional orders. Finally, using Eq. (13) in Eq. (12), the second series expansion for $\mathscr{E}_1(x,y)$ is obtained as

$$\mathscr{E}_1(x,y) = K_0(xy)-2\sum_{k=0}^{\infty}I_{k+1/2}\left[(\eta-xy)/2\right]K_{k+1/2}\left[(\eta+xy)/2\right], \tag{14}$$

The other alternative path involves, first, substitution of the series expansion of the Bessel functions of the first kind—Eq. (15)—into Eq. (12):

$$J_{2n+1}(u) = \sum_{k=0}^{\infty}\frac{(-1)^k(u/2)^{2n+1+2k}}{k!\Gamma(2n+k+2)}, \tag{15}$$

then making use of the following integral form of the modified Bessel function [23]:

$$K_v(xy) = \frac{\sqrt{\pi}}{\Gamma(v+1/2)}\left(\frac{x}{2y}\right)^{v}\int_{u=0}^{\infty}\frac{\exp\left(-x\sqrt{y^2+u^2}\right)}{\sqrt{y^2+u^2}}u^{2v}\,\mathrm{d}u, \tag{16}$$

Having done the algebraic manipulations given in Eq. (17), we obtain:

$$\begin{aligned}&2\sum_{n=0}^{\infty}\int_{u=0}^{\infty}J_{2n+1}(u)\frac{\exp\left[-x\sqrt{y^2+u^2}\right]}{\sqrt{y^2+u^2}}\,\mathrm{d}u \\ &\quad= 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\int_{u=0}^{\infty}\frac{(-1)^k(u/2)^{2n+1+2k}}{k!\Gamma(2n+k+2)}\frac{\exp\left[-y\sqrt{x^2+u^2}\right]}{\sqrt{x^2+u^2}}\,\mathrm{d}u \\ &\quad= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(n+k+1)}{k!\Gamma(k+2n+2)}\left(\frac{x}{2y}\right)^{n+k+1/2}K_{n+k+1/2}(xy),\end{aligned} \tag{17}$$

We could further simplify Eq. (17) by collecting all the terms which contain the same order modified Bessel functions as follows:

$$\begin{aligned}&= \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^k\Gamma(k+1)\left(\frac{x}{2y}\right)^{k+1/2}K_{k+1/2}(xy)\sum_{j=0}^{k}\frac{(-1)^j}{\Gamma(k+j+2)\Gamma(k-j+1)} \\ &= \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^k\Gamma(k+1)\left(\frac{x}{2y}\right)^{k+1/2}K_{k+1/2}(xy)\left\{\frac{1}{(2k+1)\Gamma^2(k+1)}\right\} \\ &= \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(2k+1)}\left(\frac{x}{2y}\right)^{k+1/2}K_{k+1/2}(xy),\end{aligned} \tag{18}$$

Finally substituting Eq. (18) into Eq. (12), we obtain the third expansion for $\mathscr{E}_1(x,y)$ as:

$$\mathscr{E}_1(x,y) = K_0(xy) - \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)k!}\left(\frac{x}{2y}\right)^{k+1/2}K_{k+1/2}(xy), \quad (y>1). \tag{19}$$

The fourth series expansion for $\mathscr{E}_1(x,y)$, which will be discussed in this article, was derived by Mamedov [20] in terms of the exponential integral functions as

$$\mathscr{E}_1(x,y) = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+1/2)}{k!}\left(\frac{xy}{\eta}\right)^{2k}E_{2k+1}(\eta), \tag{20}$$

where $E_n(x)$ is the $n$th-order exponential integral functions.

The advantages/disadvantages of these series expansions for $\mathscr{E}_1(x,y)$ will be discussed in Section 5.

Now let us turn our attention to evaluation of higher order GEIFs. In literature, the following recurrence relations could be cited for $\mathscr{E}_2(x,y)$ [11,18]:

$$\mathscr{E}_2(x,y) = \mathrm{e}^{-\eta} - x\mathscr{E}_1(x,y), \tag{21}$$

and for $n\geqslant 2$ [18]:

$$(n-1)\mathscr{E}_n(x,y) = -x\mathscr{E}_{n-1}(x,y) - x^{n-2}\frac{\partial}{\partial x}\left(\frac{\mathscr{E}_{n-2}(x,y)}{x^{n-3}}\right), \tag{22}$$

However, Eq. (22) requires the partial derivatives of GEIFs with respect to $x$ variable. Analytical evaluation of the derivatives still requires computation of a series expansion.

Our next task is to obtain a recurrence relation which, preferably, does not involve derivatives. To do this, for $n>1$, $t^{n-1}$ is expanded to binomial sum about $t=1$ as follows:

$$t^{n-1} = (1+t-1)^{n-1} = \sum_{m=0}^{n-2}\frac{(n-1)!}{m!(n-1-m)!}(t-1)^m + (t-1)^{n-1}, \tag{23}$$

After solving Eq. (23) for $(t-1)^{n-1}$, it is substituted into Eq. (2). Comparing the terms on the RHS with the integral form given by Eq. (2), we obtain:

$$\frac{\mathscr{E}_n(x,y)}{x^{n-1}} = \frac{1}{(n-1)!}\int_{t=1}^{\infty} t^{n-1}\frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}\,\mathrm{d}t - \sum_{m=0}^{n-2}\frac{1}{(n-m-1)!}\frac{\mathscr{E}_{m+1}(x,y)}{x^m}, \quad n\geqslant 2 \tag{24}$$

In order to evaluate the integral in Eq. (24), we make use of Eq. (16) as follows:

$$\int_{t=0}^{\infty}\frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}t^{n-1}\,\mathrm{d}t = \frac{\Gamma(n/2)}{\sqrt{\pi}}\left(\frac{2y}{x}\right)^{(n-1)/2}K_{(n-1)/2}(xy), \tag{25}$$

On the other hand, for integration over $0\leqslant t\leqslant 1$ interval, we will apply the same procedure used in derivation of Eq. (5). Expanding the exponential term to its Taylor series and integrating it in the prescribed interval yields:

$$\begin{aligned}\int_{t=0}^{1}\frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}t^{n-1}\,\mathrm{d}t &= \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\int_{t=0}^{1}t^{n-1}\left(\sqrt{t^2+y^2}\right)^{k-1}\mathrm{d}t\\ &= \frac{1}{ny}\sum_{k=0}^{\infty}\frac{(-xy)^k}{k!}F\left(\frac{n}{2},\frac{1-k}{2};1+\frac{n}{2};\frac{-1}{y^2}\right),\end{aligned} \tag{26}$$

Next using Eqs. (25) and (26) in Eq. (24), followed by algebraic manipulations and simplifications, a recurrence relation, for $n \geqslant 0$, in terms of the lower-order GEIFs, is obtained as

$$\mathscr{E}_{n+1}(x,y) = \frac{(xy/2)^{n/2} K_{n/2}(xy)}{\Gamma(1+n/2)} + \frac{x^{n+1}}{(n+1)!} \sum_{k=0}^{\infty} \frac{(-xy)^{k-1}}{k!} F\left(\frac{n+1}{2}, \frac{1-k}{2}; \frac{n+3}{2}; \frac{-1}{y^2}\right) - \sum_{k=1}^{n} \frac{x^k}{k!} \mathscr{E}_{n-k+1}(x,y), \tag{27}$$

In Eq. (27), for $n = 0$, the last summation term over $k$ should be ignored; thus, Eq. (7) is exactly reproduced. Similarly, for $n = 1$, Eq. (21) is generated. Repeating the recurrence relation for $n = 3$ and 5 results in the following simple relations:

$$\mathscr{E}_4(x,y) = \frac{e^{-\eta}}{3!}\left(2\eta + x^2 + 2\right) - \frac{x^3}{3!}\mathscr{E}_1(x,y) - \frac{x^2}{2!}\mathscr{E}_2(x,y) - x\mathscr{E}_3(x,y), \tag{28}$$

and

$$\mathscr{E}_6(x,y) = \frac{e^{-\eta}}{5!}\left(8\eta^2 + 4(x^2+6)\eta + x^4 + 4x^2 + 24\right) - \frac{x^5}{5!}\mathscr{E}_1(x,y) - \frac{x^4}{4!}\mathscr{E}_2(x,y) - \frac{x^3}{3!}\mathscr{E}_3(x,y) - \frac{x^2}{2!}\mathscr{E}_4(x,y) - x\mathscr{E}_5(x,y), \tag{29}$$

However, the recurrence relations for the odd orders of GEIFs, $\mathscr{E}_3(x,y)$ and $\mathscr{E}_5(x,y)$ so on, contain a series expansion which also needs to be evaluated. As the order is increased, even for the odd orders of GEIFs, the number of lower order terms increases. Nevertheless, it is possible to eliminate lower-order GEIFs in terms of only $\mathscr{E}_1(x,y)$. It is also possible to obtain a recurrence relation which involves the integral of lower-order GEIFs. The relation and its derivation is given in the Appendix section.

In order to derive a general series expansion for $\mathscr{E}_n(x,y)$, the binomial expansion of $(t-1)^{n-1}$ will be considered:

$$(t-1)^{n-1} = (n-1)! \sum_{m=0}^{n-1} \frac{(-1)^{m+n-1} t^m}{m!(n-m-1)!}, \tag{30}$$

Substituting Eq. (30) into Eq. (2) and making use of Eqs. (25) and (26), the integration with respect to $t$ variable (for $y>0$) yields:

$$\begin{aligned}\frac{\mathscr{E}_n(x,y)}{x^{n-1}} &= \frac{1}{(n-1)!}\sum_{m=0}^{n-1}\frac{(n-1)!(-1)^{m+n-1}}{m!(n-m-1)!}\int_{t=1}^{\infty} t^m \frac{\exp\left[-x\sqrt{t^2+y^2}\right]}{\sqrt{t^2+y^2}}\,dt,\\ &= \sum_{m=0}^{n-1}\frac{(-1)^{m+n-1}}{m!(n-m-1)!}\left\{\frac{\Gamma([m+1]/2)}{\sqrt{\pi}}\left(\frac{2y}{x}\right)^{m/2} K_{m/2}(xy) - \frac{1}{(m+1)y}\sum_{k=0}^{\infty}\frac{(-xy)^k}{k!}F\left(\frac{m+1}{2},\frac{1-k}{2};\frac{m+3}{2};\frac{-1}{y^2}\right)\right\},\end{aligned} \tag{31}$$

Using $\Gamma([m+1]/2) = m!\sqrt{\pi}/(2^m\Gamma(1+m/2))$ relation in Eq. (31), followed by simplifications and reorganizations of the terms, we obtain a general expression for $\mathscr{E}_n(x,y)$:

$$\mathscr{E}_n(x,y) = (-x)^{n-1}\sum_{m=0}^{n-1}\frac{(-1)^m (y/2x)^{m/2}}{\Gamma(1+m/2)(n-m-1)!}K_{m/2}(xy) + (-x)^n \sum_{m=0}^{n-1}\frac{(-1)^{m+1}}{(m+1)!(n-m-1)!} \times \sum_{k=0}^{\infty}\frac{(-xy)^{k-1}}{k!}F\left(\frac{m+1}{2},\frac{1-k}{2};\frac{m+3}{2};\frac{-1}{y^2}\right), \tag{32}$$

Setting $n = 1$ in Eq. (32), we obtain exactly Eq. (7).

## 3. Integrals of the GEI functions

In this section, the integrals of the GEIFs of interest in thermal radiative transfer will be addressed. For $a>0$, $b>0$, $v>-1$, we have the following integral identity [23]:

$$\int_{x=0}^{\infty} x^{v+1} J_v(bx) \frac{K_\mu\left(a\sqrt{x^2+y^2}\right)}{\left(\sqrt{x^2+y^2}\right)^{\mu/2}}\,\mathrm{d}x = \frac{b^v}{a^\mu}\left(\frac{\sqrt{a^2+b^2}}{y}\right)^{\mu-v-1} K_{\mu-v-1}\left(y\sqrt{a^2+b^2}\right), \tag{33}$$

Setting $\mu = 1/2$ in Eq. (33), we obtain:

$$\int_{x=0}^{\infty} x^{v+1} J_v(bx) \frac{\exp\left[-a\sqrt{x^2+t^2}\right]}{\sqrt{x^2+t^2}}\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} b^v \left(\frac{t}{\sqrt{a^2+b^2}}\right)^{v+1/2} K_{-v-\frac{1}{2}}\left(t\sqrt{a^2+b^2}\right), \tag{34}$$

Multiplying Eq. (34) with $(t-1)^{n-1}/(n-1)!$ on both sides and integrating it over $1\leqslant t<\infty$ interval, along with the use of Eq. (2), yields:

$$\begin{aligned}
\mathbb{E}_{n,v}(a,b) &= \int_{x=0}^{\infty} x^{v+1} J_v(bx) \frac{\mathscr{E}_n(a,x)}{a^{n-1}}\,\mathrm{d}x \\
&= \frac{1}{(n-1)!}\int_{x=0}^{\infty} x^{v+1} J_v(bx) \int_{t=1}^{\infty} (t-1)^{n-1} \frac{\exp\left[-a\sqrt{x^2+t^2}\right]}{\sqrt{x^2+t^2}}\,\mathrm{d}t\,\mathrm{d}x \\
&= \frac{\sqrt{2/\pi}\, b^v}{(n-1)!\left(\sqrt{a^2+b^2}\right)^{v+1/2}} \int_{t=1}^{\infty} (t-1)^{n-1} t^{v+1/2} K_{-v-\frac{1}{2}}\left(t\sqrt{a^2+b^2}\right)\mathrm{d}t,
\end{aligned} \tag{35}$$

Here, though $a^{n-1}$ term in the denominator of the left hand side (LHS) integral of Eq. (35) is a real constant, it has been kept in the LHS since radiative integral transfer kernels are in this form.

To evaluate the last integral of Eq. (35) over $t$-variable, the following integrals, for $c>0$ and $p\geqslant 1$ integer values, could be obtained if $q\neq 0,1,2,3,\ldots$,

$$\begin{aligned}
\int_{t=1}^{\infty} (t-1)^p t^q K_{-q}(ct)\,\mathrm{d}t = & \frac{-(2/c)^{p+q}}{2}\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(q+\frac{p}{2}\right){}_1F_2\left(\frac{1-p}{2};\frac{3}{2},1-q-\frac{p}{2};\frac{c^2}{4}\right) \\
& + \frac{(2/c)^{p+q}}{2c}\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(q+\frac{1+p}{2}\right){}_1F_2\left(\frac{-p}{2};\frac{1}{2},-q+\frac{1-p}{2};\frac{c^2}{4}\right) \\
& + (2c)^q\sqrt{\pi}\,\Gamma(p+1)\frac{\Gamma(-1-p-2q)}{\Gamma(1/2-q)}{}_1F_2\left(q+\frac{1}{2};1+q+\frac{p}{2},q+\frac{p+3}{2};\frac{c^2}{4}\right),
\end{aligned} \tag{36a}$$

or for integer $q = 0,1,2,\ldots$, we obtain:

$$\begin{aligned}
\int_{t=1}^{\infty} (t-1)^p t^q K_{-q}(ct)\,\mathrm{d}t = & \frac{p!}{2c}\left(\frac{2}{c}\right)^{p+q}\sum_{k=0}^{p}\frac{(-c/2)^k}{k!(p-k)!}\Gamma\left(\frac{p+1-k}{2}\right)\Gamma\left(\frac{p+2q+1-k}{2}\right) \\
& + (-1)^{p+1}\frac{p!}{2}\left(\frac{2}{c}\right)^q\sum_{k=0}^{q-1}\frac{(-1)^k(2k)!\,\Gamma(q-k)}{k!\,\Gamma(2k+p+2)}\left(\frac{c^2}{4}\right)^k \\
& + (-1)^{p+q+1}\frac{p!}{2}\left(\frac{c}{2}\right)^q\sum_{k=0}^{\infty}\frac{(c^2/4)^k\,\Gamma(2k+2q+1)}{k!\,\Gamma(k+q+1)\,\Gamma(2k+2q+p+2)} \\
& \times\{\psi(k+1)+\psi(k+q+1)-\ln(c^2/4) \\
& +2\psi(2k+2q+p+2)-2\psi(2k+2q+1)\}
\end{aligned} \tag{36b}$$

where $\psi(j+1) = -\gamma + \sum_{n=1}^{j}(1/n)$, $\gamma$ is the Euler's constant, and ${}_1F_2(\alpha;\beta,\gamma;z)$ is a generalized hypergeometric function defined as [22]:

$$ {}_1F_2(\alpha;\beta,\gamma;z) = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k+\beta)\Gamma(k+\gamma)} \frac{x^k}{k!}, \tag{37} $$

For integer $p$ ($p \geqslant 0$), the ratio $\Gamma(-1-p-2q)/\Gamma(1/2-q)$, in the last term of Eq. (36a) is indeterminate for $q = (2\ell+1)/2, \quad \ell = 0, 1, 2, \ldots$ values, and the numerical evaluation may results in warnings in some software or subprograms. In this case, the problem can be eliminated by using $\Gamma(z) = \pi csc\pi z/\Gamma(1-z)$ identity, for the numerator and denominator of the indeterminate term, followed by $L$'hospital's rule yields:

$$ \lim_{q \to (2\ell+1)/2} \frac{\Gamma(-1-p-2q)}{\Gamma(1/2-q)} = (-1)^{p+q+3/2} \frac{\Gamma(q+1/2)}{2\Gamma(p+2q+2)}, \quad \text{for } \ell = 0, 1, 2, \ldots \tag{38} $$

Now for integer $p$, $v \geqslant 0$, $n \geqslant 1$ and real $a > 0$, we can proceed with the integration of the RHS of Eq. (35), using Eqs. (36a) and (38) to obtain

$$ \begin{aligned} \mathbb{E}_{n,v}(a,b) = \int_{x=0}^{\infty} x^{v+1} J_v(bx) \frac{\mathscr{E}_n(a,x)}{a^{n-1}} \, dx = {} & \frac{(2b)^v \Gamma\big((2v+n+1)/2\big)}{\big(a^2+b^2\big)^{(2v+n+1)/2} \Gamma\big((n+1)/2\big)} \\ & \times {}_1F_2\left(\frac{1-n}{2}; \frac{1}{2}, \frac{1-n-2v}{2}; \frac{a^2+b^2}{4}\right) - \frac{(2b)^v \Gamma\big((2v+n)/2\big)}{\Gamma\big(n/2\big)\big(a^2+b^2\big)^{(2v+n)/2}} \\ & \times {}_1F_2\left(\frac{2-n}{2}; \frac{3}{2}, \frac{2-n-2v}{2}; \frac{a^2+b^2}{4}\right) + \frac{(-1)^{n+v+1}(2b)^v v!}{(2v+n+1)!} \\ & \times {}_1F_2\left(v+1; \frac{2v+n+2}{2}, \frac{2v+n+3}{2}; \frac{a^2+b^2}{4}\right), \end{aligned} \tag{39} $$

When expanded for a specified $n$ and $v$, Eq. (39) yields analytical expressions in terms of hyperbolic functions. After some tedious algebra and simplifications, the exact expressions for $v = 0$, 1, 2, 3 and 4 are obtained and presented in Table 1.

On the other hand, two-dimensional cylindrical kernels and surface integrals have the following integral form of the GEIFs:

$$ \mathscr{K}_{\alpha,n}^{v,\mu}(a,b,c) = \int_{x=0}^{\infty} x^{\alpha} J_v(ax) J_{\mu}(bx) \mathscr{E}_n(c,x) \, dx, \tag{40} $$

Next we wish to obtain exact series expansion of $\mathscr{K}_{\alpha,n}^{v,\mu}(a,b,c)$. Before proceeding with the integrations, let us recall the identities that are required in this derivation. An integral form of the modified Bessel function is given as [22]:

$$ \frac{K_v(z)}{z^v} = \frac{1}{2^{v+1}} \int_{u=0}^{\infty} \exp\left(-u - \frac{z^2}{4u}\right) \frac{du}{u^{v+1}}, \quad |\arg z| < \frac{\pi}{4} \tag{41} $$

Table 1
Analytical expansions of $\mathbb{E}_{n,v}(a,b)$ for $v = 0$, 1, 2, 3, 4 $\left(\lambda = \sqrt{a^2+b^2}\right)$

| $v$ | $\mathbb{E}_{n,v}(a,b)$ |
|---|---|
| 0 | $e^{-\lambda}/\lambda^{n+1}$ |
| 1 | $b(\lambda+n+1)\,e^{-\lambda}/\lambda^{n+3}$ |
| 2 | $b^2\big(\lambda^2+(2n+3)\lambda+(n+1)(n+3)\big)\,e^{-\lambda}/\lambda^{n+5}$ |
| 3 | $b^3\big(\lambda^3+3(n+2)\lambda^2+3(n^2+5n+5)\lambda+(n+1)(n+3)(n+5)\big)\,e^{-\lambda}/\lambda^{n+7}$ |
| 4 | $b^4\left\{\begin{array}{c}\lambda^4+2(2n+5)\lambda^3+3(2n^2+12n+15)\lambda^2 \\ +(2n+7)(2n^2+14n+15)\lambda+(n+1)(n+3)(n+5)(n+7)\end{array}\right\} e^{-\lambda}/\lambda^{n+9}$ |

Setting $v = 1/2$ in Eq. (41), the integral form for the exponential term is obtained as:

$$\frac{\exp\left(-c\sqrt{x^2+t^2}\right)}{c\sqrt{x^2+t^2}} = \sqrt{\frac{2}{\pi}}\frac{K_{1/2}\left(c\sqrt{x^2+t^2}\right)}{\left(c\sqrt{x^2+t^2}\right)^{1/2}} = \sqrt{\frac{2}{\pi}}\int_{u=0}^{\infty}\exp\left[-u-\frac{c^2\left(x^2+t^2\right)}{4u}\right](2u)^{-3/2}\,\mathrm{d}u, \tag{42}$$

Then substituting Eq. (42) into Eq. (2) gives:

$$\mathscr{E}_n(c,x) = \frac{c^n}{2\sqrt{\pi}(n-1)!}\int_{t=1}^{\infty}(t-1)^{n-1}\int_{u=0}^{\infty}\exp\left[-u-\frac{c^2(x^2+t^2)}{4u}\right]\frac{\mathrm{d}u}{u^{3/2}}\,\mathrm{d}t, \tag{43}$$

We will also make use of the following integral identity [23]:

$$\int_{x=0}^{\infty}x^{2\sigma+1}\exp\left[-\frac{c^2x^2}{4u}\right]\mathrm{d}u = \frac{1}{2}\left(\frac{4u}{c^2}\right)^{\sigma+1}\Gamma(\sigma+1), \quad \sigma > -1, \quad c^2/u > 0 \tag{44}$$

Multiplication of the Bessel functions of the integer orders of $v$ and $\mu$ with $x^\alpha$ could be expressed in terms of hypergeometric functions as [23]:

$$x^\alpha J_v(ax)J_\mu(bx) = \sum_{i=0}^{\infty}\frac{(-1)^i(ax/2)^v(bx/2)^{2i+\mu}x^\alpha}{i!(i+\mu)!v!}F\left(-i,-i-\mu;v+1;\frac{a^2}{b^2}\right), \tag{45}$$

Substituting Eqs. (43) and (45) into Eq. (40), one finds:

$$\mathscr{K}_{\alpha,n}^{v,\mu}(a,b,c) = \frac{c^n}{2\sqrt{\pi}(n-1)!}\sum_{i=1}^{\infty}\frac{(-1)^i}{i!(i+\mu)!v!}F\left(-i,-i-\mu;v+1;\frac{a^2}{b^2}\right)\int_{t=1}^{\infty}(t-1)^{n-1} \times\int_{u=0}^{\infty}u^{-3/2}\exp\left[-u-\frac{c^2t^2}{4u}\right]\int_{x=0}^{\infty}(ax/2)^v(bx/2)^{2i+\mu}x^\alpha\exp\left[-\frac{c^2x^2}{4u}\right]\mathrm{d}x\,\mathrm{d}u\,\mathrm{d}t, \tag{46}$$

For the integrals in Eq. (46), we first use Eqs. (44) for the integration over $x$ and then Eq. (41) for the integration over $u$-variable. The final integral over $t$-variable is an integral of modified Bessel function given by Eqs. (36a) or (36b) where the appropriate form depends on even or odd values of $q$ (or $\rho$). For $n \geqslant 1$, $a>0$, $b>0$, $v>-1$, $b/c<1$, $a/c<1$ and $\rho>-1$, we finally, for odd values of $\rho$, obtain

$$\begin{aligned}\mathscr{K}_{\alpha,n}^{v,\mu}(a,b,c) = {} & \frac{c^{n-1}}{4\sqrt{\pi}}\frac{(a/c)^v(2/c)^{\alpha+n}(b/c)^\mu}{(n-1)!v!}\sum_{i=0}^{\infty}\frac{(-1)^i(b/c)^{2i}}{i!(i+\mu)!} \\ & \times F\left(-i,-i-\mu;v+1;\frac{a^2}{b^2}\right)\Gamma\left(\frac{1+2i+\rho}{2}\right)\left\{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2i+n+\rho}{2}\right)\right. \\ & \times {}_1F_2\left(\frac{1-n}{2};\frac{1}{2},\frac{2-2i-n-\rho}{2};\frac{c^2}{4}\right) - c\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{2i+n+\rho-1}{2}\right) \\ & \left.\times {}_1F_2\left(\frac{2-n}{2};\frac{3}{2},\frac{3-2i-n-\rho}{2};\frac{c^2}{4}\right)\right\} + \frac{(-1)^n2^{\alpha-1}a^vb^\mu c^{n-1}}{v!} \\ & \times\sum_{i=0}^{\infty}\frac{\Gamma\left([1+2i+\rho]/2\right)^2b^{2i}}{i!(i+\mu)!(2i+n+\rho)!}F\left(-i,-i-\mu;v+1;\frac{a^2}{b^2}\right) \\ & \times {}_1F_2\left(\frac{1+2i+\rho}{2};\frac{1+2i+n+\rho}{2},\frac{2+2i+n+\rho}{2};\frac{c^2}{4}\right)\end{aligned} \tag{47a}$$

and for even $\rho$ values,

$$\mathscr{K}^{\nu,\mu}_{\alpha,n}(a,b,c) = \frac{c^{n-1}(a/c)^{\nu}(2/c)^{\alpha+n}(b/c)^{\mu}}{4\sqrt{\pi}\nu!}\sum_{i=0}^{\infty}\frac{(-1)^i(b/c)^{2i}}{i!(i+\mu)!}F\left(-i,-i-\mu;\nu+1;\frac{a^2}{b^2}\right)$$
$$\times\Gamma\left(\frac{1+2i+\rho}{2}\right)\left\{\sum_{k=0}^{n-1}\frac{(-c/2)^k}{k!(n-k-1)!}\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{2i+n+\rho-k}{2}\right)\right.$$
$$+2(-1)^n\left(\frac{c}{2}\right)^n\sum_{k=0}^{i-1+\rho/2}\frac{(-c^2/4)^k(2k)!}{k!(2k+n)!}\Gamma\left(\frac{2i-2k+\rho}{2}\right)+2(-1)^{n+i+\rho/2}\left(\frac{c}{2}\right)^{n+2i+\rho}$$
$$\times\sum_{k=0}^{\infty}\frac{(c^2/4)^k(2k+2i+\rho)!}{k!(2k+2i+n+\rho)!\Gamma(k+i+1+\rho/2)}\{\psi(k+1)+\psi(k+i+1+\rho/2)$$
$$\left.-\ln(c^2/4)+2\psi(2k+2i+n+\rho+1)-2\psi(2k+2i+\rho+1)\}\right\} \tag{47b}$$

where $\rho=\alpha+\mu+\nu$, $\psi(j+1)=-\gamma+\sum_{n=1}^{j}(1/n)$, and $\gamma$ is the Euler's constant.

## 4. Derivatives of the generalized exponential integral functions

Several relations that the GEIFs satisfy can be obtained. Taking the partial derivatives of Eq. (2) with respect to $x$ and $y$, the following relation is easily found:

$$\frac{\partial^2}{\partial x\partial y}\left(\frac{\mathscr{E}_n(x,y)}{x^{n-1}}\right)=xy\mathscr{E}_n(x,y), \tag{48}$$

Taking the first and second order partial derivatives of Eq. (2) with respect to $y$ followed by algebraic manipulations, it is also found that $\mathscr{E}_1(x,y)$ satisfies the following zeroth order modified Bessel differential equation, Eq. (49), subject to the following conditions: using Eq. (4), we immediately notice that $\mathscr{E}_1(x,0)\to\int_1^{\infty}(e^{-xt}/t)\mathrm{d}t=E_1(x)$ and $\mathscr{E}_1(x,y\to\infty)\to 0$.

$$y^2\frac{\partial^2\mathscr{E}_1}{\partial y^2}+y\frac{\partial\mathscr{E}_1}{\partial y}-x^2y^2\mathscr{E}_1(x,y)=-\frac{x^3y^2}{\eta^3}(\eta+1)e^{-\eta}, \tag{49}$$

The latter differential equation explains as to why the series expansions developed in this paper contain one of the homogeneous solution of Eq. (49); that is, $K_0(xy)$.

## 5. Results and discussions

### 5.1. Convergence of series expansions

In this article, three series expansions, Eqs. (7), (14) and (19), were derived for the computation of $\mathscr{E}_1(x,y)$. These expansions, along with Eq. (20), are compared to determine the most suitable series expansion for efficient and fast numerical computations. Another series expansion which is cited in Ref. [19] is identical to Eq. (7); thus, it is not considered to be an alternative expansion of $\mathscr{E}_1(x,y)$. The convergence criterion is chosen as $\left|\mathscr{E}_1^{(N+1)}(x,y)-\mathscr{E}_1^{(N)}(x,y)\right|/\mathscr{E}_1^{(N+1)}(x,y)<10^{-6}$ where $N$ is the number of terms used in the series expansions. The reason for selecting such a criterion is that, for $x>4$ and $y>4$, values of $\mathscr{E}_1(x,y)$ are very small, and the converged solution based solely on the general term of the series may be errenous and misleading. The values of $\mathscr{E}_1(x,y)$ for various argument sets and the number of terms that is required for convergence were computed with four series expansions and presented in Table 2. The converged values of $\mathscr{E}_1(x,y)$ are the same; therefore, it has been reported only once.

At this point, before we proceed with the convergence analysis, it is appropriate to remind that the limit value of the ratio test is a measure of the rate of decline (or shrinkage), and the convergence is affected by the

Table 2
The number of terms required for convergence of four series expansion formulas for various argument sets of $\mathscr{E}_1 x, y$

| $x$ | $y$ | $\mathscr{E}_1(x,y)$ | Eq. (7) | Eq. (14) | Eq. (19) | Eq. (20) |
|---|---|---|---|---|---|---|
| 0.1 | 0.1 | 1.820490 | 4 | 47 | | 3 |
| 0.1 | 1.0 | 1.640172 | 5 | 6 | | 15 |
| 0.1 | 2.0 | 1.361776 | 6 | 4 | 7 | 31 |
| 0.1 | 3.0 | 1.131180 | 6 | 3 | 5 | 59 |
| 0.1 | 5.0 | 0.804303 | 7 | 3 | 3 | 128 |
| 0.1 | 10.0 | 0.384358 | 9 | 2 | 3 | 367 |
| 1.0 | 0.1 | 0.218099 | 10 | 57 | | 3 |
| 1.0 | 1.0 | 0.135547 | 12 | 8 | | 15 |
| 1.0 | 2.0 | 0.053515 | 15 | 5 | 9 | 36 |
| 1.0 | 3.0 | 0.019275 | 19 | 4 | 6 | 67 |
| 1.0 | 5.0 | 0.002396 | 26 | 4 | 5 | 152 |
| 1.0 | 10.0 | 1.355E−5 | 44 | 3 | 4 | 456 |
| 2.0 | 0.1 | 0.048379 | 14 | 64 | | 4 |
| 2.0 | 1.0 | 0.020156 | 17 | 9 | | 14 |
| 2.0 | 2.0 | 0.003548 | 24 | 6 | 10 | 37 |
| 2.0 | 3.0 | 0.000512 | 31 | 5 | 7 | 71 |
| 2.0 | 5.0 | 9.321E−6 | 45 | 4 | 5 | 161 |
| 2.0 | 10.0 | 3.75E−10 | 79 | 3 | 4 | 490 |
| 5.0 | 0.1 | 0.001119 | 21 | 81 | | 3 |
| 5.0 | 1.0 | 1.390E−4 | 33 | 11 | | 6 |
| 5.0 | 2.0 | 2.135E−6 | 48 | 7 | 14 | 40 |
| 5.0 | 3.0 | 1.953E−8 | 65 | 6 | 10 | 76 |
| 5.0 | 5.0 | 1.09E−12 | 90 | 5 | 7 | 152 |
| 5.0 | 10.0 | 1.63E−23 | 145 | 1 | 2 | 587 |

rate of decline of the terms in a series. Thus, the smaller the rate of decline ($a_{k+1}/a_k \to 0$), the faster (with less number of terms) the series will converge.

In Table 2, it is observed that Eq. (7), for small $x$ and $y$ values, converges with relatively few terms. As $x$ and $y$ increase, the number of terms required for convergence also increases. The reason behind this is trend can be best understood upon employing a convergence test. Applying the ratio test to the general term of the series of Eq. (7) yields,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{|xy|F(1/2,-k/2,3/2,-1/y^2)}{(k+1)F(1/2,1-k/2,3/2,-1/y^2)} \to \frac{|xy|}{(k+1)}\sqrt{1+\frac{1}{y^2}} \to \frac{\eta}{k+1} \to 0, \tag{50}$$

Eq. (50) indicates that this series expansion is unconditionally convergent regardless of $\eta$ values; however, the number of terms required for the convergence will proportionally increase as $\eta$ increases.

It is also observed that, for very small $x$ and $y$ values, the convergence of Eq. (20) is achived with relatively few terms. But, especially, for increasing $y$, the number of terms that is required for convergence also increases. Again resorting to the ratio test, and for large $n$ using $E_n(x) \to e^{-x}/(x+n)$ [22], we arrive at the following criterion:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(2k+1)(2k+1+\eta)}{(2k+2)(2k+3+\eta)}\left(\frac{y^2}{1+y^2}\right) \to \frac{y^2}{1+y^2} < 1, \tag{51}$$

Eq. (51) confirms that this series expansion is also unconditionally convergent. However, the rate of decline is dependent on $y$ and is independent of indice $k$. So as $y$ increases, the rate of decline approaches to unity, thereby increasing the number of terms required for the convergence. When the limit value approaches 1, the series sum oscillates near the correct value, and very accurate solution cannot be obtained unless the convergence criterion is increased. In this case, the accuracy of the series sum is compromised. Overall convergence performance of Eq. (7) is better than that of Eq. (20) for a wide range of argument combinations.

When the convergence behavior of Eqs. (14) and (19) are examined, it is clear that, for $x>1$ and $y>2$, not only they do better than other expansion formulas, but also very accurate solutions are obtained with only a few terms. For very large values of $x$ and $y$, only one or two terms become sufficient. In order to get more insight on the convergence nature of these two series expansions, we employ the ratio test to Eqs. (14) and (19), respectively, as follows:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{I_{k+3/2}([\eta-xy]/2)K_{k+3/2}([\eta+xy]/2)}{I_{k+1/2}([\eta-xy]/2)K_{k+1/2}([\eta+xy]/2)} \to \left(\sqrt{1+y^2}-y\right)^2 < 1, \tag{52}$$

and

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(2k+1)(x/y)}{(2k+2)(2k+3)}\frac{K_{k+3/2}(xy)}{K_{k+1/2}(xy)} \to \frac{(2k+1)(x/y)}{(2k+2)(2k+3)}\frac{2k}{xy} \to \frac{1}{y^2} < 1, \tag{53}$$

Eqs. (52) and (53) reveal the nature of the numerical conclusion observed in Table 2. Here the rates of decline, for both expansions, are also functions of $y$ only. Although Eq. (14) is unconditionally convergent, as $y\to 0$, the ratio of decline will approach unity, leading slow or oscillating convergence behavior. In Table 2, it is obvious that using Eq. (14), as $y\to 0$, the number of terms required for convergence also increases. However, as $y\to\infty$, the rate of decline approaches to zero, leading convergence with only very few terms. On the other hand, the series expansion given by Eq. (19) is divergent for $y<1$; thus the number of terms for $\mathcal{E}_1(x,y)$ corresponding to these cases are not listed in Table 2. Similar to Eq. (52), as $y$ increases, Eq. (53) approaches zero, yielding convergence with a few terms. Therefore, these two series expansion formulas are very good candidates to be used as asymptotic expansions of $\mathcal{E}_1(x,y)$

The series expansion for the $n$th order GEIFs, Eq. (32), was tested against data presented in Refs. [20,21]. The results for various orders of GEIFs and argument sets are in excellent agreement with those of Refs. [20,21], matching all significant decimal places reported. Thus to save space and avoid numerical repetition, the results with Eq. (32) have not been reported here. However, from the convergence point of view, a word is in order. Applying the ratio test to the general terms of infinite sums, it becomes evident that the convergence behavior of Eq. (7) and Eq. (32), and similarly, Eqs. (20) and (23) of Ref. [21] are identical. Therefore, previous discussions and comparisons on Eqs. (7) and (20) sheds light on the convergence behaviour of the series expansions of $\mathcal{E}_n(x,y)$. From the computational point of view, it should be pointed out that, although the hypergeometric functions, Eq. (6), which appear in Eqs. (7) and (32) are represented by an infinite sum, they yield a finite sum for negative integer values of either $\alpha$ or $\beta$, or to be exact, the summation will have $1+\min(-\alpha,-\beta)$ terms. Therefore, the evaluation of the hypergeometric functions in the expansion formulas become computationally less involved. To illustrate the magnitude of the computation time, the cpu times of each series expansion have also been investigated. For arguments $(x, y) = (0.1, 10)$, $(0.1, 2)$, $(1, 2)$, $(2, 10)$ and $(5, 2)$, the cpu times of $\mathcal{E}_1(x,y)$ yield, $78\times10^{-5}$, $63\times10^{-5}$, $141\times10^{-5}$, $892\times10^{-5}$ and $484\times10^{-5}$ s with Eq. (7) and 0.189, $734\times10^{-5}$, $921\times10^{-5}$, 0.3058 and $719\times10^{-5}$ s with Eq. (20), respectively. It is clear in $\mathcal{E}_1(5,2)$ case that, even though Eq. (7) converges to the solution with 48 while Eq. (20) with 40 terms, the cpu time using Eq. (7) is considerably less. On the other hand, for arguments $(x,y) = (0.1, 10)$ and $(2, 10)$, the cpu time of $\mathcal{E}_1(x,y)$ yields, $31\times10^{-5}$ and $34\times10^{-5}$ s with Eq. (14) and $19\times10^{-5}$ and $32\times10^{-5}$ s with Eq. (19).

### *5.2. Recurrence relations*

Besides the recurrence relation for $\mathcal{E}_2(x,y)$, Eq. (21), a more general recurrence relation was obtained—Eq. (27). The series term of Eq. (27) yields elementary functions for the even orders of the GEIFs, Eqs. (21), (28) and (29). For the odd orders of the GEIFs, the series expansion cannot be simplified in terms of elementary functions, and it needs to be evaluated. One drawback of this recurrence relation maybe cited as the order of GEIFs increase, the number of lower order GEIFs which need to be evaluated also increases. In cases where very high order of GEIFs are to be evaluted, from the computational efficiency point of view, using the general expansion formula for $\mathcal{E}_n(x,y)$, Eq. (32), would be more convenient.

### 5.3. Integrals of the GEIFs

Exact series expansions for some of the integrals of GEIFs, Eqs. (39) and (40), which are of interest in thermal radiative transfer, were derived. Hypergeometric functions in the series expansion of $\mathbb{E}_{n,v}(a,b)$, positive integer $v$'s, yield exponential functions. Thus, the exact expressions for $\mathbb{E}_{n,v}(a,b)$'s, for $v=0$, 1, 2, 3 and 4, are listed in Table 1. Radiative integral transfer kernels and the incident radiation contribution terms from walls of the two- dimensional cylindrical medium have the form of $\mathscr{K}_{\alpha,n}^{v,\mu}(a,b,c)$ function. The series expansion formulas for $\mathscr{K}_{\alpha,n}^{v,\mu}(a,b,c)$, which are traditionally evaluated by means of numerical integrations, were derived for arbitrary $v$, $\mu$, $\alpha$ and $n$. For some $v$, $\mu$, $\alpha$ and $n$ indices and for $(a, b, c)$ argument combinations, the $\mathscr{K}_{\alpha,n}^{v,\mu}(a,b,c)$ functions are computed accordingly (depending on odd or even values of $\rho$) with Eqs. (47a) and

Table 3
The exact values and absolute errors of $\mathscr{K}_{\alpha,n}^{v,\mu}(a,b,c)$ for various argument and indice combinations

| Arguments | | | Exact | Error | Exact | Error |
|---|---|---|---|---|---|---|
| $a$ | $b$ | $c$ | Eq. (47a) | | Eq. (47b) | |
| $n=1$ | | | $(\alpha, v, \mu)=(1, 0, 0)$ | | $(\alpha, v, \mu)=(1, 1, 0)$ | |
| 0.0 | 0.1 | 1.0 | 3.62425E-1 | 1.E-6 | 0 | 0 |
| 0.1 | 0.5 | 1.0 | 2.59827E-1 | 0 | 1.91528E-2 | 1.E-7 |
| 0.1 | 1.0 | 3.0 | 4.22558E-3 | 1.E-8 | 1.56265E-4 | 0 |
| 0.5 | 1.0 | 2.0 | 2.03863E-2 | 0 | 4.23689E-3 | 1.E-8 |
| 1.5 | 2.0 | 5.0 | 1.39353E-4 | −1.E-9 | 4.85020E-5 | 0 |
| | | | $(\alpha, v, \mu)=(2, 1, 0)$ | | $(\alpha, v, \mu)=(2, 1, 1)$ | |
| 0.0 | 0.1 | 1.0 | 1.10534E-5 | 0 | 0 | 0 |
| 0.1 | 0.5 | 1.0 | 3.42272E-2 | 2.E-7 | 4.77225E-2 | 0 |
| 0.1 | 1.0 | 3.0 | 1.46747E-4 | −1.E-9 | 1.45753E-4 | 0 |
| 0.5 | 1.0 | 2.0 | 4.01199E-3 | 0 | 6.48273E-3 | 0 |
| 1.5 | 2.0 | 5.0 | 2.51438E-5 | −2.E-10 | 3.82810E-5 | 0 |
| | | | $(\alpha, v, \mu)=(1, 1, 1)$ | | $(\alpha, v, \mu)=(0, 1, 1)$ | |
| 0.0 | 0.1 | 1.0 | 5.39149E-7 | −1.E-10 | 0 | 0 |
| 0.1 | 0.5 | 1.0 | 1.61024E-2 | 0 | 6.72728E-3 | 1.E-8 |
| 0.5 | 1.0 | 2.0 | 4.13428E-3 | 0 | 3.18116E-3 | 0 |
| 1.5 | 2.0 | 5.0 | 4.83737E-5 | −3.E-10 | 7.25710E-5 | 0 |
| $n=2$ | | | $(\alpha, v, \mu)=(1, 0, 0)$ | | $(\alpha, v, \mu)=(1, 1, 0)$ | |
| 0.2 | 0.5 | 1.0 | 2.27416E-1 | 0 | 3.64681E-2 | 0 |
| 0.5 | 1.2 | 2.5 | 7.09599E-3 | 0 | 1.26999E-3 | 0 |
| 1.0 | 1.5 | 4.0 | 6.49659E-4 | 0 | 1.93059E-4 | 0 |
| | | | $(\alpha, v, \mu)=(2, 1, 0)$ | | $(\alpha, v, \mu)=(2, 0, 0)$ | |
| 0.2 | 0.5 | 1.0 | 6.60588E-2 | 1.E-7 | 3.60654E-1 | 0 |
| 0.5 | 1.2 | 2.5 | 9.43859E-4 | 0 | 5.05251E-3 | 1.E-8 |
| 1.0 | 1.5 | 4.0 | 3.28776E-5 | 0 | 3.64229E-4 | 1.E-9 |
| $n=3$ | | | $(\alpha, v, \mu)=(1, 0, 0)$ | | $(\alpha, v, \mu)=(1, 1, 0)$ | |
| 0.2 | 0.5 | 1.0 | 2.03670E-1 | 0 | 3.38350E-2 | 1.E-7 |
| 0.5 | 1.2 | 2.5 | 6.42809E-3 | 0 | 1.14786E-3 | 0 |
| 1.0 | 1.5 | 4.0 | 6.08202E-4 | 0 | 1.85036E-4 | 0 |
| $n=4$ | | | $(\alpha, v, \mu)=(1, 0, 0)$ | | $(\alpha, v, \mu)=(0, 0, 0)$ | |
| 0.2 | 0.5 | 1.0 | 1.82940E-1 | 1.E-6 | 1.58356E-1 | 1.E-7 |
| 0.5 | 1.2 | 2.5 | 5.83874E-3 | 0 | 1.14917E-2 | 0 |
| 1.0 | 1.5 | 4.0 | 5.70944E-4 | 0 | 1.48269E-3 | 0 |
| $n=5$ | | | $(\alpha, v, \mu)=(1, 0, 0)$ | | $(\alpha, v, \mu)=(1, 1, 0)$ | |
| 0.2 | 0.5 | 1.0 | 1.64792E-1 | 0 | 2.69868E-2 | −1.E-7 |
| 0.5 | 1.2 | 2.5 | 5.31736E-3 | 0 | 8.91022E-4 | 0 |
| 1.0 | 1.5 | 4.0 | 5.37381E-4 | 0 | 1.66881E-4 | 0 |

(47b) and tabulated in Table 3, along with the absolute errors computed against those of numerical integrations. It should be noted that $\mathscr{K}^{\nu,\mu}_{\alpha,n}(a,b,c) = \mathscr{K}^{\nu,\mu}_{\alpha,n}(b,a,c)$ symmetry could be used to extend the comparison data provided in Table 3. The numerical integration results were obtained using an adaptive quadrature algorithm and the convergence criteria for both methods was set as $\zeta < 10^{-9}$. The agreement between the numerical and exact formulations is excellent. The $\mathscr{K}^{\nu,\mu}_{\alpha,n}(a,b,c)$ kernels also converge rapidly with a few terms when $a/c$ and $b/c$ ratios are very small. The cpu time comparison of the exact series and numerical integrations indicate that the usage of exact series expressions is superior to numerical computations. For example, for $\mathscr{K}^{0,0}_{1,1}(0.5,1,2)$, $\mathscr{K}^{1,1}_{1,1}(1.5,2,5)$ and $\mathscr{K}^{1,1}_{2,2}(1,1.5,4)$, the cpu times for the exact and numerical computations are, respectively, $46\times10^{-5}$, 0.172; $62\times10^{-5}$, 0.203 and $47\times10^{-5}$, 0.218 s. The $\mathscr{K}^{\nu,\mu}_{\alpha,n}(a,b,c)$ functions display identical convergence behavior as that of Eq. (7), and they are conditionally convergent—$b/c<1$, $a/c<1$.

## 6. Conclusion

In this paper, three series expansion formulas for $\mathscr{E}_1(x,y)$ and a general expansion formula for the $n$th order GEIFs, $\mathscr{E}_n(x,y)$, were derived. The convergence performance of the series expansions were investigated numerically and theoretically. Very efficient algorithms are presented for small and/or large arguments which are suitable for numerical computations. It was found that all three series expansion formulas could be computationally exploited where their performance is the best. The general expansion formula for the GEIFs converges very rapidly for moderate and small arguments. For very large arguments, the recurrence relations coupled with Eqs. (14) or (19) is very advantageous and could be considered as asymptotic expansion formulas. Efficient series expansions for general $\mathscr{K}^{\nu,\mu}_{\alpha,n}(a,b,c)$ kernels and spatial moments of GEIFs have been derived. It was found that cpu-time wise the exact series formulas are very economical, requiring much less computation time. Exploiting efficient programming tools and techniques could also drastically reduce the computation time of the any of the series expansion formulas given in this paper. The recurrence and derivative properties, and analytical solutions for integrals of interest to thermal radiative researchers have also been generated and computationally tested.

## Appendix

In this section, the exact series expansions for some of the integrals of GEIFs, which may be useful for researchers in the radiative transfer field, have been presented.

To evaluate the $n$th order integral of GEIF over $0 \leqslant y < \infty$ interval, using Eqs. (2) and (3), and recalling the following identity [22]:

$$\frac{Ki_n(x)}{x^n} = \frac{1}{(n-1)!}\int_{t=1}^{\infty}(t-1)^{n-1}K_0(xt)\,\mathrm{d}t, \tag{A.1}$$

where $Ki_n(x)$ is the $n$th order Bickley-Naylor functions, we easily obtain

$$\int_{y=0}^{\infty}\mathscr{E}_n(x,y)\,\mathrm{d}y = \frac{Ki_n(x)}{x}. \tag{A.2}$$

Similarly multiplying Eq. (2) by $y$ and integrating it in $0 \leqslant y < \infty$ interval, using integration by parts, yields,

$$\int_{y=0}^{\infty}y\mathscr{E}_n(x,y)\,\mathrm{d}y = \frac{\mathrm{e}^{-x}}{x^2}, \tag{A.3}$$

It is interesting to see that the integral is independent of order $n$. Similarly, using integration by parts, one can obtain the following integrals as well:

$$\int_{y=0}^{\infty}y^3\mathscr{E}_n(x,y)\,\mathrm{d}y = \frac{2(x+n+1)}{x^4}\mathrm{e}^{-x} \tag{A.4}$$

and

$$\int_{y=0}^{\infty} y^5 \mathscr{E}_n(x,y)\,\mathrm{d}y = \frac{2^3}{x^6}\left(x^2+(2n+3)x+(n+1)(n+3)\right)\mathrm{e}^{-x} \tag{A.5}$$

We can generalize the integrals involving multiplication of the GEIFs with $y^m$. The nature of the outcome obliges us to treat the integrations seperately for even and odd powers of $y$. The integration steps involve basically multiplication of Eq. (32) by $y^{2m+1}$ or $x^m$, and integrating over $(0,\infty)$ interval [22,23]. In this section, skipping the lengthy steps, the following series expansions (for $m = 0,1,2,..$) are reported:

$$\begin{aligned}\int_{y=0}^{\infty} y^{2m+1}\mathscr{E}_n(x,y)\,\mathrm{d}y = m!\left(\frac{4}{x^2}\right)^m \Bigg\{ & \frac{(-1)^{m+n+1}x^{2m+n-1}m!}{(2m+n+1)!} \\ & \times {}_1F_2\left(m+1;\frac{2m+n+2}{2},\frac{2m+n+3}{2};\frac{x^2}{4}\right) \\ & +\frac{\Gamma\left((2m+n+1)/2\right)}{x^2\Gamma\left((n+1)/2\right)}\,{}_1F_2\left(\frac{1-n}{2};\frac{1}{2},\frac{1-n-2m}{2};\frac{x^2}{4}\right) \\ & -\frac{\Gamma\left((2m+n)/2\right)}{x\Gamma\left(n/2\right)}\,{}_1F_2\left(\frac{2-n}{2};\frac{3}{2},\frac{2-n-2m}{2};\frac{x^2}{4}\right)\Bigg\}\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}\int_{y=0}^{\infty} y^{2m}\mathscr{E}_n(x,y)\,\mathrm{d}y = & \frac{(-x)^{n-1}}{\sqrt{\pi}}\left(\frac{4}{x^2}\right)^m\Gamma\left(m+\frac{1}{2}\right)\sum_{k=0}^{n-1}\frac{(-1)^k}{k!(n-k-1)!} \\ & \times\left\{\begin{array}{l}\frac{1}{2x}\left(\frac{2}{x}\right)^k\Gamma\left(\frac{2m+k+1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)-\frac{1}{2}\sum\limits_{j=0}^{m-1}\frac{(-1)^j(m-j-1)!}{j!(2j+k+1)}\left(\frac{x^2}{4}\right)^j \\ +\left(\frac{x^2}{4}\right)^m\sum\limits_{j=0}^{\infty}\frac{(-1)^{m+1}(x^2/4)^j}{j!(m+j)!(2j+2m+k+1)^2} \\ +\frac{1}{2}\left(\frac{x^2}{4}\right)^m\sum\limits_{j=0}^{\infty}\frac{(-1)^{m+1}(x^2/4)^j\left\{\psi(j+1)+\psi(j+m+1)-\ln(x^2/4)\right\}}{j!(m+j)!(2j+2m+k+1)}\end{array}\right\},\end{aligned} \tag{A.7}$$

where $\psi(j+1) = -\gamma+\sum_{n=1}^{j}(1/n)$ and $\gamma$ is the Euler's constant. For some cases, in equation (A.6), indeterminate situations may arise, and they should be treated as described in Section 3. On the other hand, the integrals of ($\mathscr{E}_n$ $(x,y)$ involving $x^m$ is obtained rather easily by using Eq. (2) and integrating it over $x$-variable first,

$$\begin{aligned}\int_{x=0}^{\infty} x^m\mathscr{E}_n(x,y)\,\mathrm{d}x &= \frac{1}{(n-1)!}\int_{t=1}^{\infty}\frac{(t-1)^{n-1}}{\sqrt{t^2+y^2}}\int_{x=0}^{\infty}x^{m+n-1}\exp\left(-x\sqrt{t^2+y^2}\right)\mathrm{d}x\,\mathrm{d}t \\ &= \frac{(m+n-1)!}{(n-1)!}\int_{t=1}^{\infty}(t-1)^{n-1}(t^2+y^2)^{-(m+n+1)/2}\,\mathrm{d}t\end{aligned} \tag{A.8}$$

Finally performing the integration over $t$-variable [23], for $y\geqslant 0$, we obtain:

$$\int_{x=0}^{\infty} x^m\mathscr{E}_n(x,y)\,\mathrm{d}x = \frac{m!}{m+n}F\left(\frac{m+1}{2},\frac{m+2}{2};\frac{m+n+2}{2};-y^2\right),\quad (m\geqslant 0,\quad n\geqslant 1) \tag{A.9}$$

A recurrence relation involving integral of GEIF can be derived as well. Let us consider Eq. (1) for $\mathscr{E}_n(xu,y/u)$ and integrate this over $1\leqslant u<\infty$:

$$\int_{u=1}^{\infty}\mathscr{E}_n(xu,y/u)\,\mathrm{d}u = \frac{x^{n-1}}{(n-1)!}\int_{u=1}^{\infty}\int_{t=xu}^{\infty}(u-1)^{n-1}\frac{\exp\left[-\sqrt{t^2+(xy)^2}\right]}{\sqrt{t^2+(xy)^2}}\,\mathrm{d}t\,\mathrm{d}u, \tag{A.10}$$

It is noted that the exponential term of Eq. (A.10) is not a function of $u$ variable, and so integration over $u$-variable can be performed first. Upon reorganizing the integration intervals of Eq. (A.10), we can write

$$\int_{u=1}^{\infty} \mathscr{E}_n(xu, y/u)\,\mathrm{d}u = \frac{x^{n-1}}{(n-1)!}\int_{t=x}^{\infty} \frac{\exp\left[-\sqrt{t^2+(xy)^2}\right]}{\sqrt{t^2+(xy)^2}} \int_{u=1}^{t/x} (u-1)^{n-1}\,\mathrm{d}u\,\mathrm{d}t,$$

$$= \frac{x^{-1}}{n!}\int_{t=x}^{\infty} (t-x)^n \frac{\exp\left[-\sqrt{t^2+(xy)^2}\right]}{\sqrt{t^2+(xy)^2}}\,\mathrm{d}t. \tag{A.11}$$

Inspection of the last integral with respect to Eq. (1), it is noticed that the integral is the GEIF of order $n+1$, and for $n \geqslant 0$, so we deduce the following recursive integral relation:

$$\mathscr{E}_{n+1}(x,y) = x\int_{u=1}^{\infty} \mathscr{E}_n(xu, y/u)\,\mathrm{d}u = x\int_{u=0}^{1} \mathscr{E}_n(x/u, yu)\frac{\mathrm{d}u}{u^2} \tag{A.12}$$

## References


[1] Tables of the generalized exponential integral functions, Harvard University, vol. 31. Cambridge, MA: Harvard University Press; 1949.

[2] Chandrasekhar S. On the diffuse reflection of a pencil of radiation by a plane parallel atmosphere. Proc Natl Acad Sci 1958;44:933–40.

[3] Chandrasekhar S. Radiative transfer. New York: Dover Publications; 1960.

[4] Bellman R, Kalaba R, Ueno S. On the diffuse reflection of parallel rays by an inhomogeneous flat layer as a limiting process. J Math Anal Appl 1963;7:91–9.

[5] Kourganoff V. Basic methods in transfer problems. New York: Dover Publications; 1963.

[6] Goody RM. Atmospheric radiation. London: Oxford University Press; 1964.

[7] Smith GM. The transport equation with plane symmetry and isotropic scattering. Proc Cambridge Philos Soc 1964;60:909–21.

[8] Hunt GE. The transport equation of radiative transfer in a three-dimensional space with anisotropic scattering. J Inst Math Appl 1967;3:181–92.

[9] Rybicki GB. The searchlight problem with isotropic scattering. J Quant Spectrosc Radiat Transfer 1971;11:827–49.

[10] Oberoi RS, Callaway J, Seiler GJ. Analytic evaluation of integrals in variational calculations of scattering theory. J Comput Phys 1972;10:466–74.

[11] Breig WF, Crosbie AL. Numerical computation of a generalized exponential integral function. Math Comput 1974;28:575–9.

[12] Breig WF, Crosbie AL. Two-dimensional radiative equilibrium. J Math Anal Appl 1974;46:104–25.

[13] Crosbie AL, Dougherty RL. Two-dimensional radiative transfer in a cylindrical geometry with anisotropic scattering. J Quant Spectrosc Radiat Transfer 1981;25:551–69.

[14] Dertsine KL, Bareiss EH. Generation of neutron transport benchmark problems using BEAPAC-3 T. Prog Nucl Energy 1981;8:309–18.

[15] Drummond JE. The anharmonic oscillator: perturbation series for cubic and quartic energy distortion. J Phys A: Math Gen 1981;14:1651–61.

[16] Crosbie AL, Lee LC. Relation between multidimensional radiative transfer in cylindrical and rectangular coordinates with anisotropic scattering. J Quant Spectrosc Radiat Transfer 1987;38:231–41.

[17] Crosbie AL, Lee LC. Multidimensional radiative transfer: a single integral representation of anisotropic scattering kernels. J Quant Spectrosc Radiat Transfer 1989;42:239–46.

[18] Yuen WW, Wong LW. Numerical computation of an important integral function in two-dimensional radiative transfer. J Quant Spectrosc Radiat Transfer 1983;29:145–9.

[19] Altaç Z. Integrals involving Bickley and Bessel functions in radiative transfer, and generalized exponential integral functions. ASME J. Heat Transfer 1996;118:789–92.

[20] Mamedov BA. Evaluation of generalized exponential integral function using binomial expansion theorems. J Quant Spectrosc Radiat Transfer 2005;94:507–14.

[21] Guseinov II, Mamedov BA. On the evaluation of generalized exponential integral functions. J Quant Spectrosc Radiat Transfer 2006;102:251–6.

[22] Abramowitz M, Stegun IA. Handbook of mathematical functions. UK: Dover Publications Inc.; 1972.

[23] Gradshteyn IS, Ryzhik IM. Table of integrals, series, and products. USA: Academic Press; 1980.